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THERMAL RESONANCE IN THE HEAT CONDUCTION PROBLEM WITH A HEAT SOURCE MOVING ALONG A CLOSED CIRCUIT

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We found the conditions of resonant increase in the amplitude of temperature oscillations of the points of a thin heat-conducting rod in the shape of a closed ring with a variable-power heat source moving on its surface.

In [1] it is shown that the transfer of a liquid coolant in a closed circulation circuit, containing an immovable active zone with a periodically varying heat flux, involves a resonance phenomenon consisting of a sharp increase in the amplitude of temperature oscillations for a definite relationship among the oscillation frequency of the heat flux in the active zone, the rate of coolant transfer, and the circuit length. In effect, this phenomenon is similar to the resonance in mechanical oscillatory systems with coincidence of the natural frequencies and the frequencies of disturbing processes. In the problem treated in [1], the natural frequency depends on the convective term in the heat transfer equation that defines the time of liquid circulation in the circuit and on the boundary condition of circuit closure that is tantamount to periodicity of the solution with respect to the coordinate running along the circuit.

It may be assumed that, when similar conditions hold, resonance can also occur in other heat problems, including the problem of heat conduction of solids in the presence of moving sources with the heat release $q(x-ut, t)$. However, here the problem must have specific features of its own. We now consider them, taking, as a case in point, a thin heat-conducting rod of length L in the shape of a closed ring with a heat source moving on its surface. The simplest realization of such a ring is a wheel rim, during whose motion over the bearing surface heat release occurs at the contact point. With a small thickness of the ring, a one-dimensional formulation of the problem appears to be possible:

$$\frac{\partial T}{\partial t} - a\frac{\partial^2 T}{\partial x^2} = q\,(x - ut,\ t)\,. \tag{1}$$

We introduce new variables $\xi = x\text{-}ut$ and $\tau = t$, thereby converting to a coordinate system that is stationary relative to the source, and then Eq. (1) yields the equation

$$\frac{\partial T}{\partial \tau} - u\frac{\partial T}{\partial \xi} - a\frac{\partial^2 T}{\partial \xi^2} = q\,(\xi\,,\ \tau)\,. \tag{2}$$

This equation differs from that examined in [1] by the presence of the second derivative of T and the need to take into account the effect of the thermal diffusivity, and it should be given special consideration.

With a nonstationary heat flux $q(\xi, \tau) = q_s(\xi) + q_0(\xi) \exp (i\omega\tau)$, the solution to Eq. (2) is sought in the form

$$T\,(\xi\,,\ \tau) = T_s\,(\xi) + \vartheta\,(\xi) \exp\,(i\omega\tau)\,. \tag{3}$$

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Substituting Eq. (3) into Eq. (2) and introducing the dimensionless quantity $\xi = \xi/L$ (the symbol of dimensionlessness is hereinafter omitted) instead of ξ, for the amplitude of the nonstationary temperature component we obtain the equation

$$\frac{d^2\vartheta}{\partial\xi^2} + 2\sqrt{\alpha}\,\frac{d\vartheta}{d\xi} - i\beta\vartheta = -Q_0(\xi)\,, \tag{4}$$

where

$$\alpha = \left(\frac{uL}{2a}\right)^2, \quad \beta = \frac{\omega L^2}{a}, \quad q_0 = \frac{q_0 L^2}{a}.$$

To solve Eq. (4), as in [1], we use the condition of solution continuity along the closed circuit rather than boundary conditions; however, in contrast to [1], not only must the function $\vartheta(\xi)$ be continuous but also its derivative $d\vartheta/d\xi$. A general solution $\vartheta(\xi)$ can be found for arbitrary functions $Q_0(\xi)$. The present study, however, treats only a single heat flux related to the vicinity of the point $\xi = \xi^*$:

$$Q_0(\xi) = \begin{cases} 0 & \text{for} \quad 0 < \xi < \xi^*, \quad \xi^* + \Delta\xi < \xi < 1\,, \\ \dfrac{A}{\Delta\xi} & \text{for} \quad \xi^* \le \xi \le \xi^* + \Delta\xi\,, \quad \Delta\xi \to 0\,. \end{cases} \tag{5}$$

In this case Eq. (4) admits a solution in elementary functions. Based on this solution and taking, without loss of generality, the point $\xi = 0$ as the observable one, after simple manipulations we derive the following expressions for the temperature amplitude:

$$|\vartheta| = \frac{A}{2\sqrt{\alpha^2 + \beta^2}}\Big\{ [C_1 \exp(-\lambda_1 \xi^*) + C_2 \exp(-\lambda_2 \xi^*)]^2 +$$

$$+ [D_1 \exp(-\lambda_1 \xi^*) + D_2 \exp(-\lambda_2 \xi^*)]^2\Big\}^{1/2}, \tag{6}$$

$$C_k = \frac{(-1)^{k+1}\,[\mu\cos\lambda\xi^* + (-1)^k \lambda \sin\lambda\xi^*]\,a_k - [\lambda\cos\lambda\xi^* + \mu(-1)^{k+1}\sin\lambda\xi^*]\,b_k}{1 - 2\cos\lambda\exp(-\lambda_k) + \exp(-2\lambda_k)}\,, \tag{7}$$

$$D_k = \frac{(-1)^k\,[\lambda\cos\lambda\xi^* + \mu(-1)^{k+1}\sin\lambda\xi^*]\,a_k + [\mu\cos\lambda\xi^* + \lambda(-1)^k\sin\lambda\xi^*]\,b_k}{1 - 2\cos\lambda\exp(-\lambda_k) + \exp(-2\lambda_k)}\,, \tag{8}$$

$$\lambda_k = (-1)^{k+1}\mu - \sqrt{\alpha}\,, \quad k = 1\,,\,2\,;$$

$$a_k = 1 - \cos\lambda\exp(-\lambda_k)\,; \quad b_k = \sin\lambda\exp(-\lambda_k)\,; \tag{9}$$

$$\mu = \sqrt{\left(\frac{(\alpha^2+\beta^2)^{1/2} + \alpha}{2}\right)}\,, \quad \lambda = \sqrt{\left(\frac{(\alpha^2+\beta^2)^{1/2} - \alpha}{2}\right)}\,. \tag{10}$$

Using the above expressions we analyze the dependences of $|\vartheta|$ on the velocity u of motion of the source, its position on the circuit ξ^*, the oscillation frequency ω of the heat flux, and the thermal diffusivity a. These dependences are ultimately governed by the parameters α and β in Eq. (4) or, accordingly, by the parameters λ and μ.

It should be pointed out that, with constant a, u, and L, the parameters λ and μ are uniquely related to ω; making allowance for $\beta = \omega L^2/a$ and $\alpha = (uL/2a)^2$ and converting expression (10) with respect to ω we arrive at

$$\omega = \frac{2a\lambda}{L^2}\sqrt{\left(\lambda^2 + \left(\frac{uL}{2a}\right)^2\right)}, \quad \omega = \frac{2\mu a}{L^2}\sqrt{\left(\mu^2 - \left(\frac{uL}{2a}\right)^2\right)}. \tag{11}$$

It is seen from Eq. (11) that for a change in $\omega \in (0, \infty)$ there are corresponding changes in $\lambda \in (0, \infty)$ and $\mu \in (uL/2a, \infty)$.

For a fixed source ($\alpha = 0$) with coincident observation and source points ($\xi^* = 0$ or 1), having regard for $\mu = \lambda = \sqrt{\beta/2} = \sqrt{\omega L^2/2a}$, after not uncomplicated rearrangements of Eqs. (6)-(10) we obtain a simpler expression for the amplitude:

$$|\vartheta| = \frac{A}{4\lambda}\left(\frac{\operatorname{ch} 2\lambda - \cos 2\lambda}{2 - 4\cos\lambda \operatorname{ch}\lambda + \operatorname{ch} 2\lambda + \cos 2\lambda}\right)^{1/2}. \tag{12}$$

It follows from Eq. (12) that, with an increase in λ corresponding to an increase in ω, the amplitude $|\vartheta|$ monotonically decreases exponentially without any singularities.

In the case of a moving source ($\alpha \neq 0$), however, the dependence of $|\vartheta|$ on ω in not trivial. Let us assess this dependence, bearing in mind that the thermal diffusivity for ordinary technical materials is relatively small, amounting to ~ 1 cm^2/sec. Then, with a source velocity u of ~ 1 m/sec and a circuit length L of ~ 1 m, α may be considered large. Setting $\alpha >> \beta$, based on Eqs. (6)-(10) we obtain the following expression for the amplitude at $\xi^* \neq 0$ (1):

$$|\vartheta| \approx \frac{A}{\sqrt{\alpha}} \exp\left(-\frac{\lambda^2 \xi^*}{2\sqrt{\alpha}}\right) \frac{|\cos\lambda\xi^* + \sin\lambda\xi^*|}{\sqrt{\left(1 - 2\cos\lambda \exp\left(-\frac{\lambda^2}{2\sqrt{\alpha}}\right) + \exp\left(-\frac{\lambda^2}{\sqrt{\alpha}}\right)\right)}}. \tag{13}$$

In the above expression, the function under the radical in the denominator is important. At values of λ close to $2\pi n$ and large values of α, the function becomes small in accordance with the estimate $\lambda^2/2\alpha$, and $|\vartheta|$ increases, assuming values determined by the equation

$$|\vartheta|_{\max} = \frac{A}{\lambda^2} \exp\left(-\frac{\lambda^2 \xi^*}{2\sqrt{\alpha}}\right) \left|\cos\lambda\xi^* + \sin\lambda\xi^*\right|, \quad \lambda = 2\pi n. \tag{14}$$

The indicated values $\lambda = 2\pi n$, at which the exponential decrease in $|\vartheta|$ with increasing λ is interrupted, may be called resonance ones. At these values of λ, in conformity with Eq. (11), a resonance relation between the oscillation frequency of the heat flux of the source and the velocity of its motion is obtained:

$$\frac{\omega_0 L^2}{2a} = 2\pi n \sqrt{(2\pi n)^2 + (uL/2a)^2}. \tag{15}$$

When $a \to 0$, the simpler resonance condition $u = \omega L/2\pi n$ follows from Eq. (5), which exactly coincides with the resonance relation in [1]. This is a natural result because, at $a = 0$, the effect of the second derivative in Eq. (2) vanishes and the problem transforms to that examined in [1]. In the general case with $a \neq 0$, the resonance conditions should be refined. Specifically, in accordance with Eq. (15), at $a \neq 0$ the resonance frequency ω_0 itself is larger than at $a = 0$. With constant u and a, ω_0 decreases with increasing L, but more slightly than at $a = 0$.

All these results are the case with large values of $\alpha >> \beta$.

We performed a numerical analysis of relations (6)-(10) for arbitrary values of α and β, and its results are given in Fig. 1. It is seen that the resonance sharpness at $\lambda = 2\pi$ rises with increasing α but $|\vartheta|_{\max}$ at the resonance

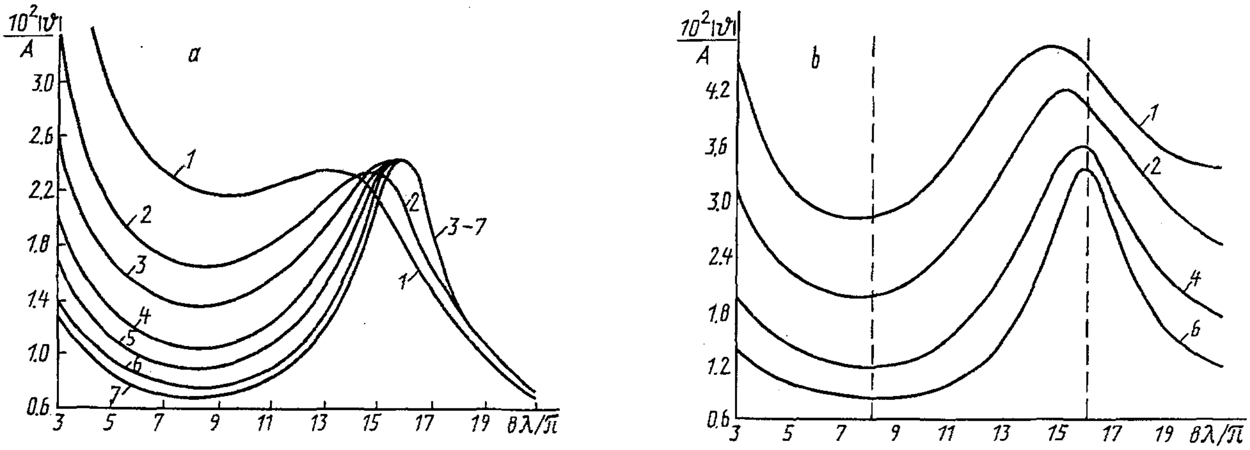

Fig. 1. Relative amplitude of the temperature oscillations $|\vartheta|$ A vs the reduced oscillation frequency λ of the heat flux from the source and the square of the reduced velocity α of it movement: a) $\xi^* = 0.5$, b) $\xi^* = 0$ (1); 1) $\alpha = 100$; 2) 200; 3) 300; 4) 400; 5) 700; 6) 1000; 7) 1200.

point at values of ξ^* sufficiently far from the observation point ($\xi^* \neq 0$, $\xi^* \neq 1$) is practically independent of α and, at $\xi^* = 0.5$, is equal to ~ 0.025 A, according to Eq. (14). At the same time, the minimal values $|\vartheta|_{min}$, which occur at $\lambda = \pi$, are strongly dependent on α.

The dependences of $|\vartheta|$ on $\lambda(\omega)$ at $\xi^* = 0$ (1) are characterized by certain distinctive features (Fig. 1b). Firstly, the values of $|\vartheta|$ are larger than at intermediate values of ξ^* and, secondly, not only $|\vartheta|_{min}$ but also $|\vartheta|_{max}$ depend noticeably on α. The general character of the dependences of $|\vartheta|_{max}$ and $|\vartheta|_{min}$ on ξ^* at a fixed α is evident from the curves in Fig. 2a; initially, as the source point ξ^* recedes from the observation point $\xi = 0$, $|\vartheta|_{max}$ and $|\vartheta|_{min}$ decrease almost linearly and thereafter, as the points approach each other on the opposite side ($\xi^* = 1$), they increase abruptly. Mathematically, such nonsymmetry of the dependences in linked with nonsymmetry of the coefficients C_k and D_k in Eq. (6), which invalidates approximation (13) at $\xi^* = 1$.

At any fixed values of ξ^*, a unique dependence of $|\vartheta|_{min}/|\vartheta|_{max}$ on α can be constructed. Figure 2b exemplifies such dependence for $\xi^* = 0.5$; it is a monotonically descending curve, since with increasing α the minimal value is $|\vartheta|_{min} \to 0$.

It seems possible to utilize the obtained results in analyzing the temperature fields and the corresponding strength characteristics of wheels with a heat-conducting rim, moving over a wavy bearing surface with the profile $y_s = y_{so} + h \sin(2\pi x_s/l)$. For this profile, at the contact point there is a variable normal load that at small h is defined by the equation:

$$N = mg\left\{1 - \pi^2\left(\frac{h}{l}\right)^2 [1 + \cos(4\pi x_s/l)]\right\} \tag{16}$$

and, hence, the heat release is $E = 2f_r\mathrm{Nu}/D$. With account for $x_s = ut$, the variable component of the heat release is $\Delta E = 2\pi^2(h/l)^2 f_r(mgu/D)\cos(4\pi ut/l)$. Since in Eq. (2) $q = \Delta E/\rho C_p S\Delta x$, with the dimensionlessness adopted above in Eq. (4) we obtain $A = 2\pi^2(h/l)^2 f_r(mguL/DS_x)$ and $\omega = 4\pi u/l$. Substituting ω into Eq. (15) and remembering that in this case $L = \pi D$, we find the resonance relation

$$\frac{\pi^2 uD^2}{al} = \sqrt{\left(4\pi^2 + \left(\frac{\pi Du}{2a}\right)^2\right)}\ , \tag{17}$$

connecting the wheel diameter D, the reduced velocity of wheel movement Du/a, and the waviness length of the bearing surface l. When $a \to 0$, relation (17) simplifies, taking the form $l = 2\pi D$. Thus, at small a, we draw the practically important conclusion that the conditions of resonant increase in the temperature depend weakly on the velocity u; the latter affects only the rise in the temperature amplitude in accordance with the value of A.

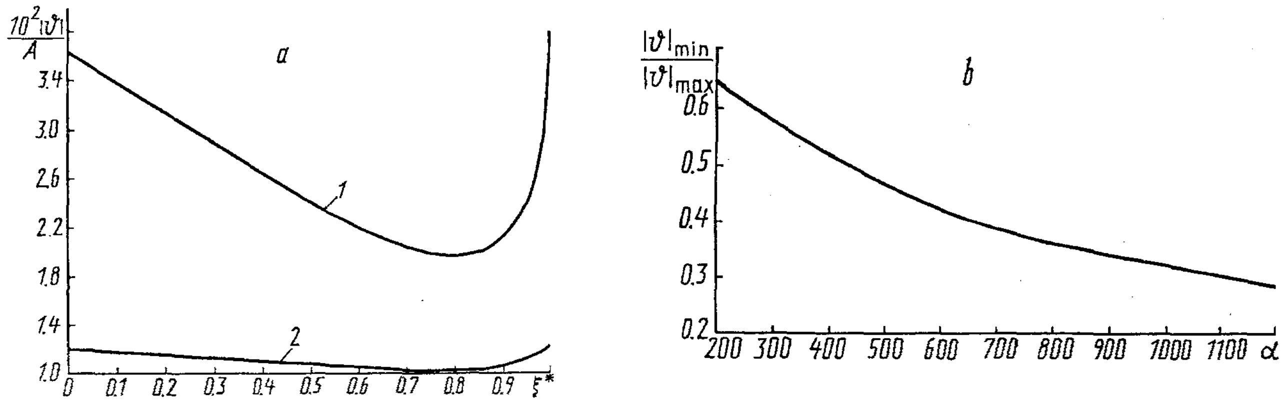

Fig. 2. Maximal $|\vartheta|_{max}$ and minimal $|\vartheta|_{min}$ amplitudes of the temperature oscillation as functions of the source position ξ^* on the circuit for $\alpha = 500$ (a) and the ratio $|\vartheta|_{min}/|\vartheta|_{max}$ as a function of α for $\xi^* = 0.5$ (b). a:1) $|\vartheta|_{max}/A$, 2) $|\vartheta|_{min}/A$.

The results obtained can also be applied to measuring facilities for remote noncontact measurement of rotation frequencies and thermal diffusivities of the materials of rotating bodies. A measurement procedure with a periodic source is the following. Use is made of a concentrated energy source (e.g., a laser beam) with an obturator that permits a change in the radiation power with a controllable frequency. The beam is directed to the chosen point of the ring ξ^*. The temperature at a preset point of the ring ($\xi = 0$) is measured with the aid of thermovision equipment. In measurements, the frequency $\omega = \omega^*$ is selected at which the maximal value $|\vartheta|_{max}$ is provided, and the rotation velocity $u \approx \omega^* L/2\pi$ is determined. Next, the rotation frequency is reduced by half ($\omega^{**} = \omega^*/2$) and $|\vartheta|_{min}$ is measured, the ratio $|\vartheta|_{min}/|\vartheta|_{max}$ is calculated, and from a plot of the type of that presented in Fig. 2b the parameter α and, accordingly, the thermal diffusivity $a = uL/2\sqrt{\alpha}$ are obtained.

In conclusion we note that the resonance effect found also occurs in other problems of heat conduction with a moving heat source. Preliminary evaluations of the temperature fields in finite-thickness rings and solid disks performed on the basis of the heat conduction equation in a cylindrical system of coordinates demonstrate that the resonance problem ultimately also reduces to analyzing a one-dimensional equation of the type of Eq. (2) with all ensuing results. Therefore, it seems possible to extend the range of applicability of the above method for measuring the rotation frequency and the parameter a to heat-conducting bodies of various shapes.

NOTATION

T, ϑ, temperature and its oscillation frequency; $a = \chi/c\rho$, χ, c, ρ, thermal diffusivity, thermal conductivity, specific heat, and density of the rod material; q, q_0, $Q_0 = q_0 L^2/a$, heat flux, its oscillation frequency, and dimensionless oscillation amplitude; x, coordinate running along the rod; L, circuit length; t, τ, time; $\xi = x\text{-}ut$, coordinate in the moving reference system; u, source velocity; ω, ω_0, oscillation frequency and its resonance value; $A = Q_0\Delta\xi$, source parameter; $\Delta\xi$, circuit section with the source; ξ^*, coordinate of the source point; $\alpha = (uL/2a)^2$, square of the reduced velocity of the source; $\beta = \omega L^2/a$, dimensionless oscillation frequency; λ, λ_1, λ_2, μ, parameters that are functions of ω, L, u, and a; $|\vartheta|$, amplitude of temperature oscillations at the observation point; C_1, C_2, D_1, D_2, constants in solving Eq. (4); S, cross section of the rod; y_s, x_s, coordinates of the bearing surface; h, l, height and length of the wave of the bearing surface; m, wheel mass; g, acceleration due to gravity; D, wheel diameter; f_r, rolling friction.

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